Points, Curves, Surfaces

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Roadmap

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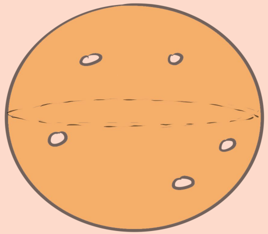
01

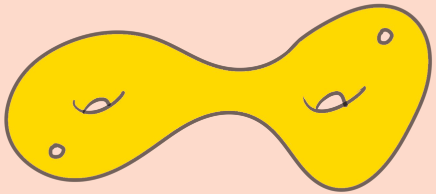
Background

Surfaces

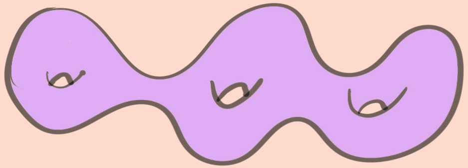
We consider finite type surfaces with negative Euler characteristic (Topologically, genus g surfaces with n points removed)



Sphere with puncture



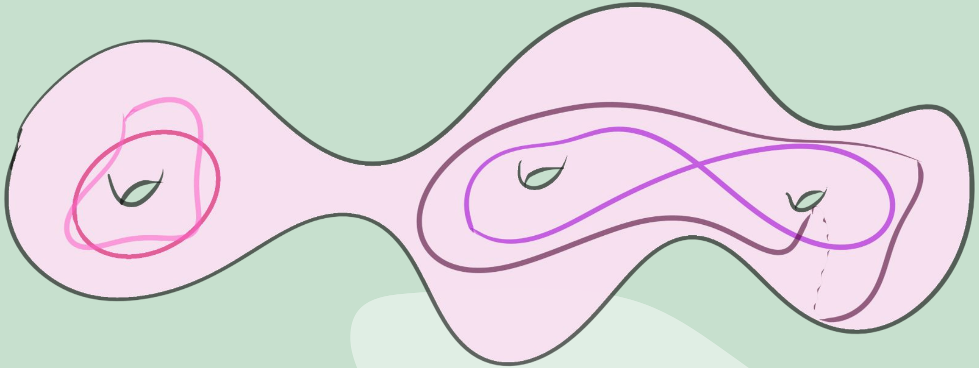
Surface with genus and puncture



Closed surfaces

Curves

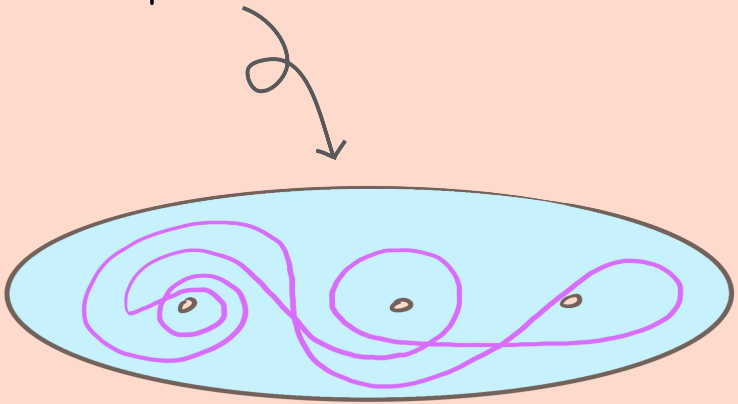
- Topology: We consider all curves up to homotopy class
- Geometry: In each homotopy class there is a unique geodesic (shortest one)



Filling curves

A closed curve on a surface is said to be filling if it intersects every non-trivial simple closed curve on the surface.

Example:

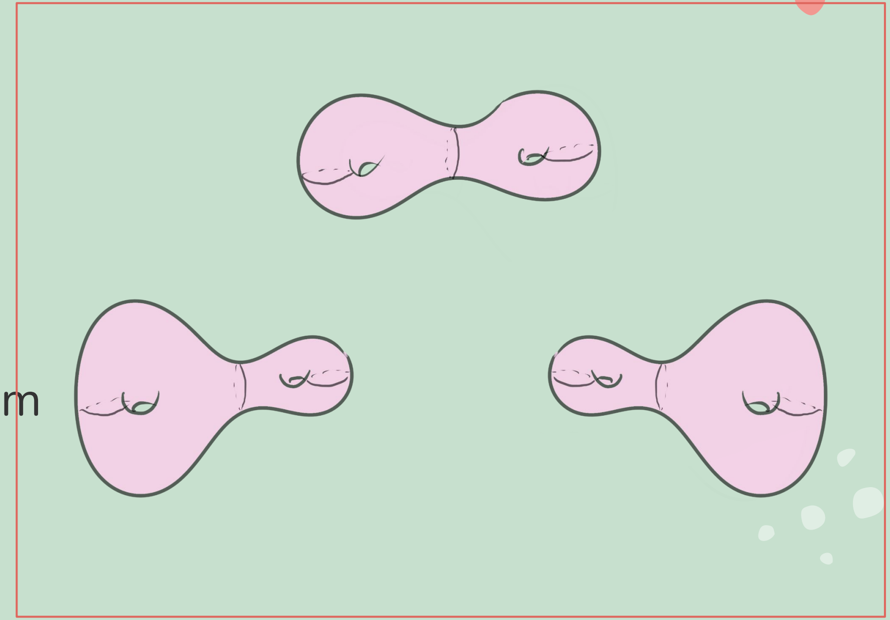


Complement of a filling curve is a union of discs and annuli

Teichmüller space

Set of isotopy classes of marked hyperbolic structures.

Each point in Teichmuller space of Σ, can be denoted as (X,f) where X is a surface with complete, finite area hyperbolic structure with geodesic boundary and f is a diffeomorphism from Σ to X.



(X, f) ~ (Y, g) if $f \circ g^{-1}$ is an isometry.

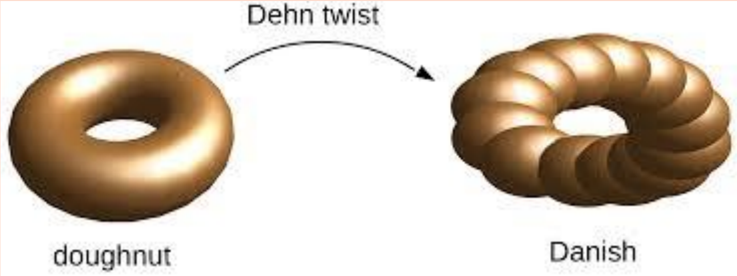
Mapping Class Group

Group of orientation preserving isometries up to isotopy

MCG := Diffeo $^{+}$ (Σ) / ~

f ~ g if $f \circ g^{-1}$ is isotopic to identity

Generated by Dehn twists



Action of MCG

The MCG acts naturally on the Teichmuller space.

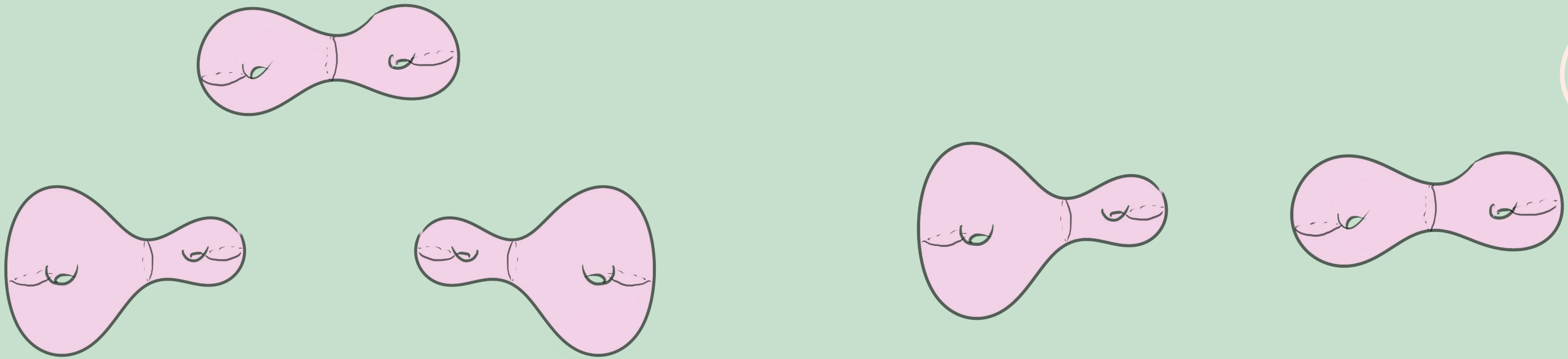
$$g \circ (X,f) \longrightarrow (X, f \circ g^{-1})$$

Acts by "unmarking"

Moduli space

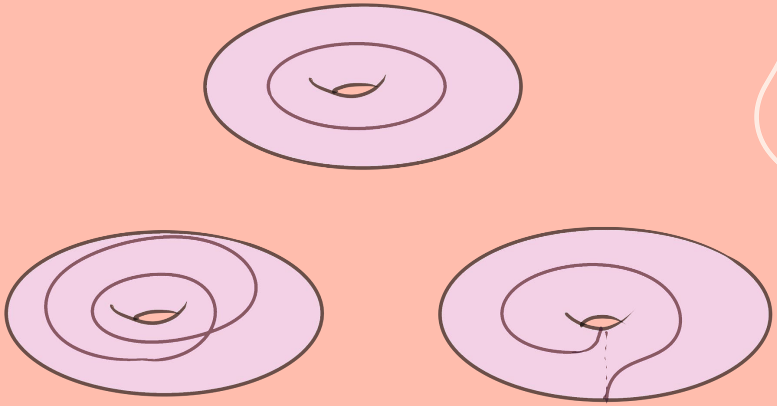
Moduli space Mg is the quotient of Teichmuller space under the action of MCG.

Two points (X, f) and (X, g) that map to the same point in moduli space differ by the action of the mapping class $g^{-1} \circ f$



Topological Types for curves

Two curves are said to be of the same topological type if there is a mapping class group element taking one to the other.



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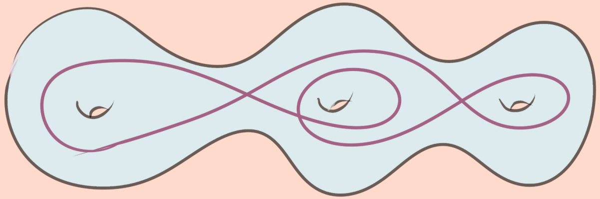
MCG Invariants of a curve

Self-Intersection number

If γ non-simple closed curve in Σ , then the self intersection number of γ denoted by $i(\gamma, \gamma)$ is the minimum number of self-intersection points a curve in its free homotopy class has.



• Min-invariant

Fix a topological surface Σ and let Teich(Σ) denote its Teichmuller space.
Consider a non-simple closed curve **γ** in Σ.

For (**ϕ**,**X**) in(Σ) . Let $l_\gamma(X)$ denote the 'X-length' of the geodesic in the free **homotopy class of ϕ(γ)**.

We define the length infimum of **γ** as follows:

$$m_\gamma = \inf\{\, l_\gamma(X) : (\phi, X) \text{ in Teich}(\Sigma)\}$$

Properties of min invariant

- Invariant under action on Mapping Class Group on Teichmuller space.

- The infimum is attained

- The infimum is unique.

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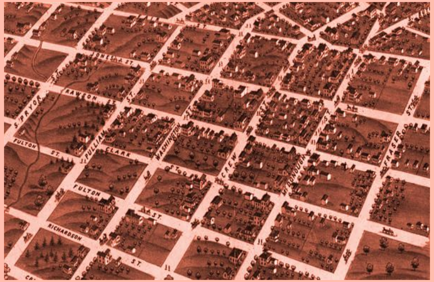
Relation between both invariants

Questions ?

1. Are any of these complete invariants? Can they distinguish topological types?
2. Can one invariant distinguish curves that the others can't?
3. Are there relations between these invariant? Like growth rates etc?
4. What are these relations?

Why?

Self intersection number vs Length



Want to buy : Self intersections

Currency: Length

04

Results

Main Theorems

Theorem 5.2 (Separating curve construction). *Suppose $\chi(\Sigma) \leq -2$. There exists an infinite set of positive integer $\mathcal{K}$ and a collection of curve pairs $\{(\alpha_k, \beta_k)\}_{k\in\mathcal{K}}$ so that*

(1) *α_k and β_k are each filling curves*
(2) $i(\alpha_k, \alpha_k) = i(\beta_k, \beta_k) = k$
(3) $m_{\alpha_k} \lesssim \log k$ *and* $m_{\beta_k} \gtrsim \sqrt{k}$
(4) $sys(X_{\beta_k}) \geq 2r\left(\frac{\ell_{\gamma_0}(X_{\gamma_0})}{2}\right)$. *In particular, the metrics $\{X_{\beta_k}\}_{k\in\mathcal{K}}$ are contained in a compact subspace of $\mathcal{M}$.*
(5) *Make systole statement about the metrics X_{α_k}. In particular, $X_{\alpha_k} \to Y \in \partial\mathcal{M}$, where Y is the degenerate metric attained by letting the length of η go to zero.*

Main Theorems

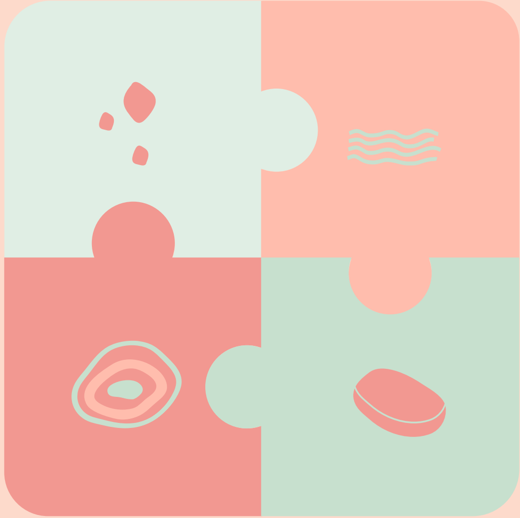
Theorem 5.3. *[Punctured surface case] Suppose Σ has negative Euler characteristic with genus g and $n \geq 1$ punctures. There exists an infinite set of positive integer $\mathcal{K}$ and a collection of curve pairs $\{(\alpha_k, \beta_k)\}_{k \in \mathcal{K}}$ so that*

(1) *α_k and β_k are each filling curves*
(2) *$i(\alpha_k, \alpha_k) = i(\beta_k, \beta_k) = k$*
(3) *$m_{\alpha_k} < m_{\beta_k}$*
(4) *the metrics X_{α_k} and X_{β_k} stay within a compact subspace of $\mathcal{M}$.*

Proof ingredients

Families of filling curves

Designer metrics

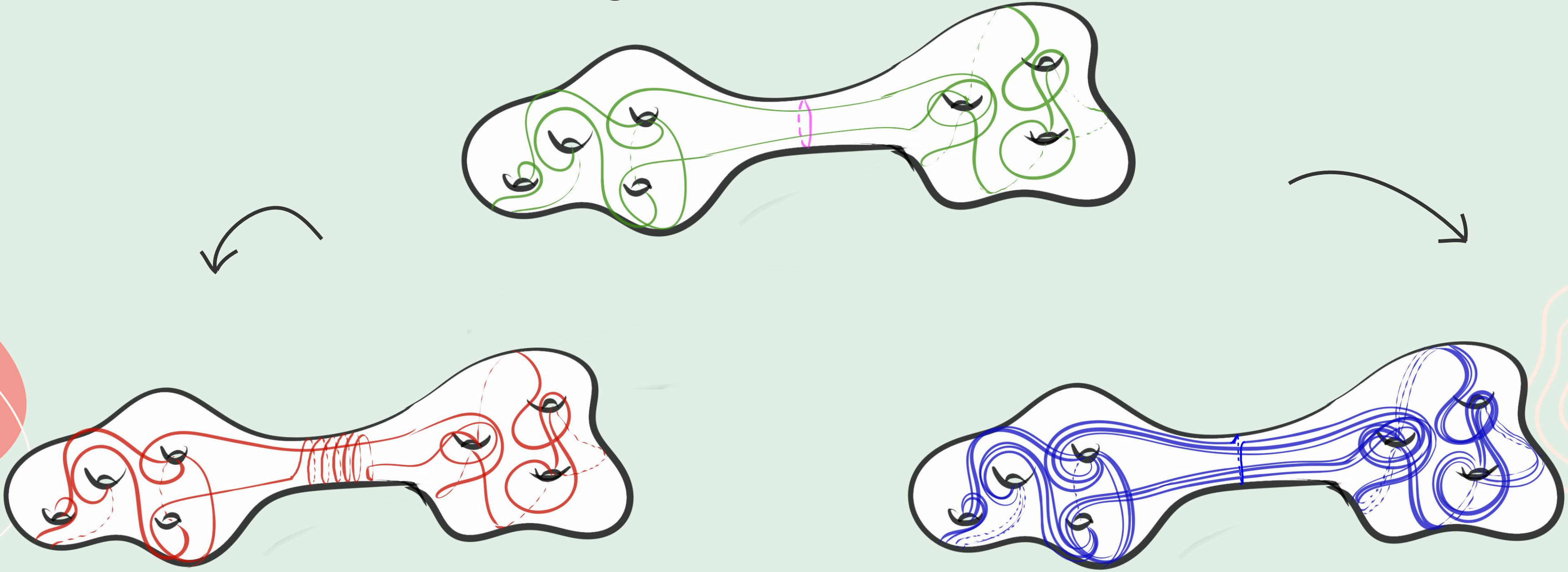


Length bounds

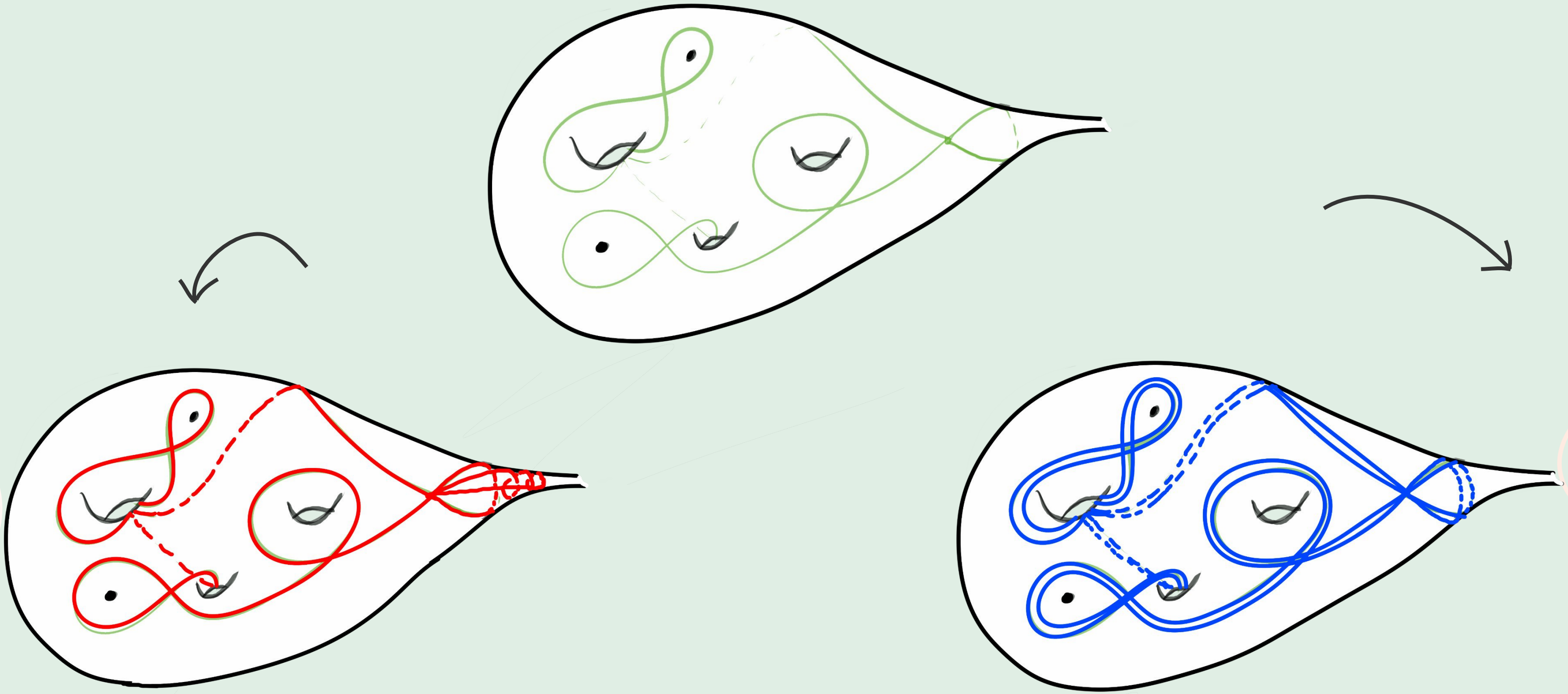
Minimal filling curves

Proof sketch

Case 1 : Separating case



Case 2 : Cusp case



Other results/Corollaries

- Coarse min lengths for these two families of curves.
- Specify the metrics that achieve the min length.
- These detect topological type that self intersection number cannot.

Thanks!

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